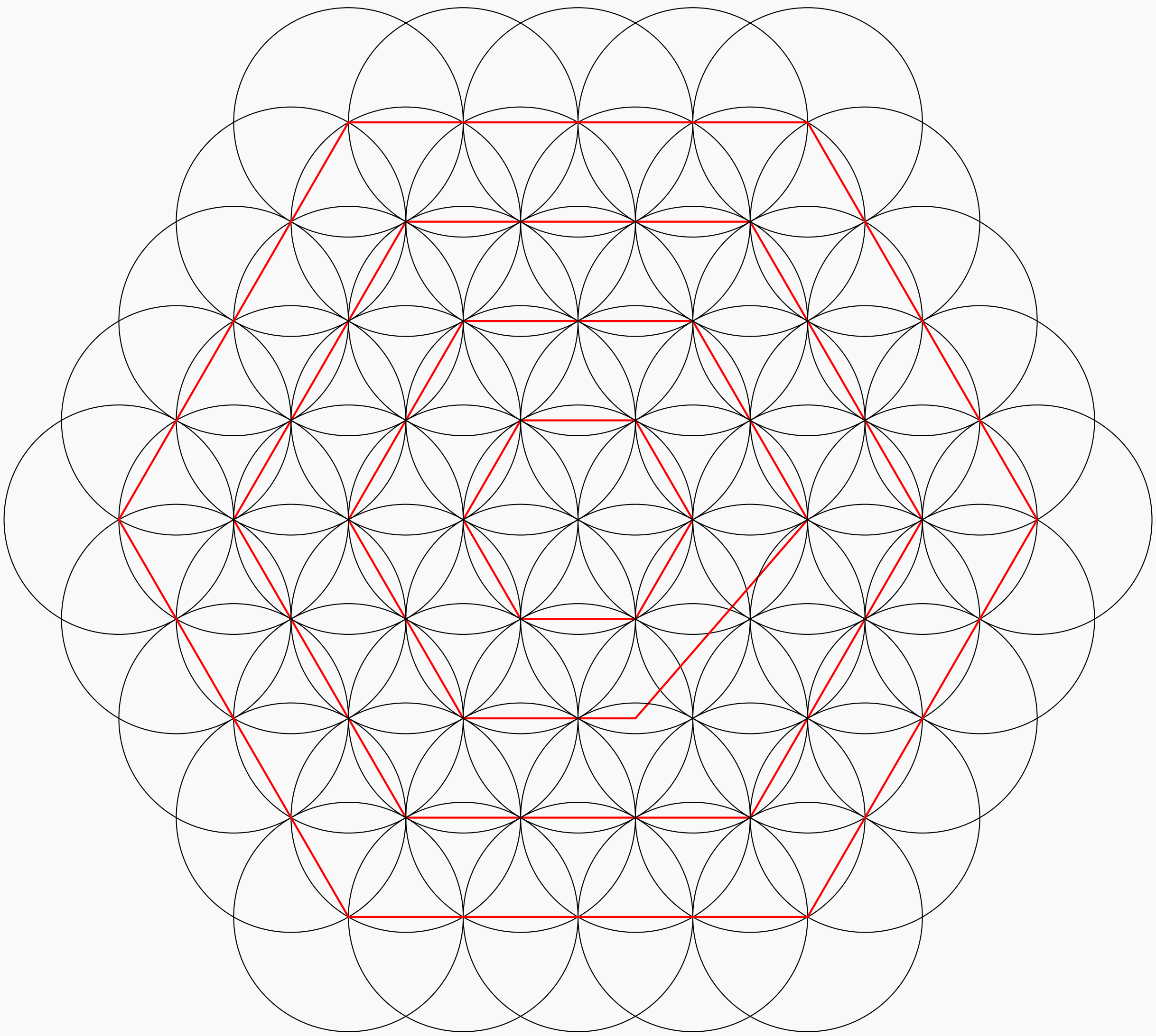
Grade 6 Math Circles

Nov 15/16/17, 2022

Geometric Constructions - Problem Set Solutions

1. Construct a repeating geometric pattern of your choice. (You have the freedom to decide, bonus points if the pattern looks nice, or if you discover a new geometric property)

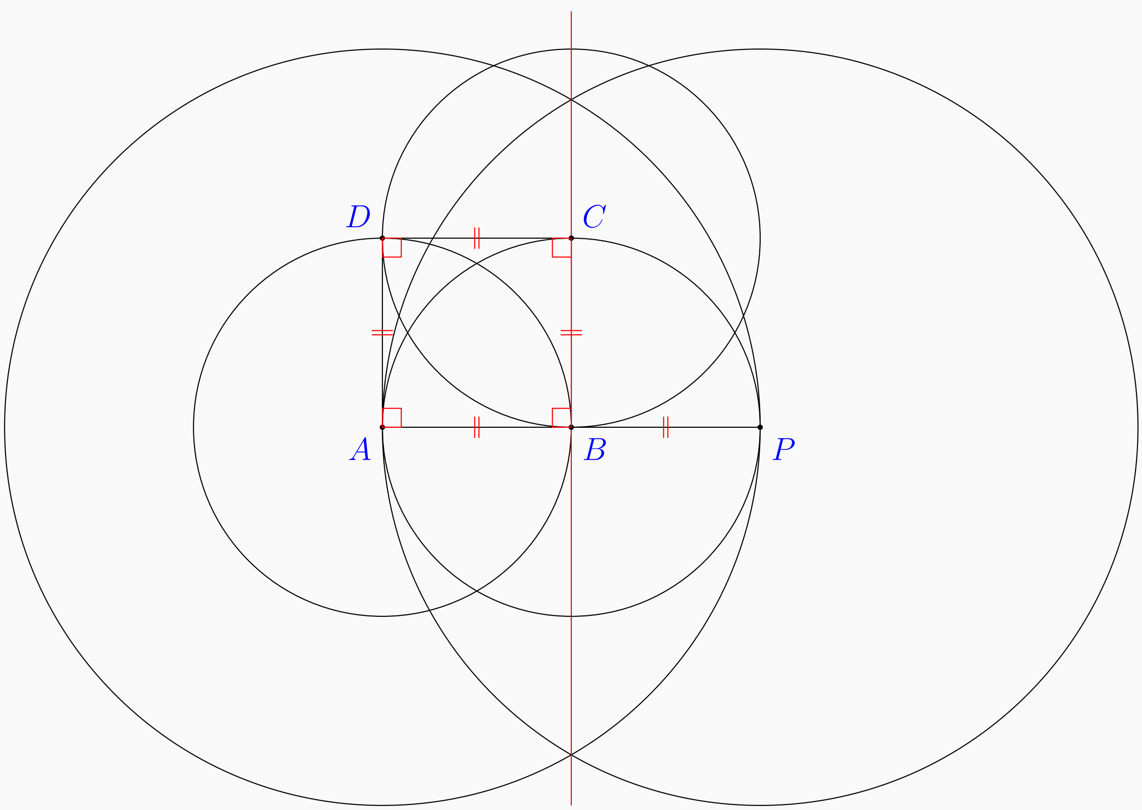
Solution: Responses may vary. For example, if we take inspiration from Example 1 (the regular hexagon), we could get



2. Construct a square.

Solution: The simplest approach is to construct a square using the construction of a perpendicular bisector. Here is a possible summary of the steps to create a square $ABCD$:

1. Draw two points A and P, and construct the perpendicular bisector of $\overline{AP}$.
2. Let B be the midpoint of $\overline{AP}$. Construct a circle with center B and radius AB. Let C be one of the points where the perpendicular bisector of $\overline{AP}$ intersects this circle.
3. Construct a circle with center C and radius BC. Construct a circle with center A and radius AB.
4. Let D be the second intersection of these two circles (the first is B). Then $ABCD$ is a square with side length $CD = BC = AB = AD$.



3. Draw a line ℓ and a point P (not on ℓ).

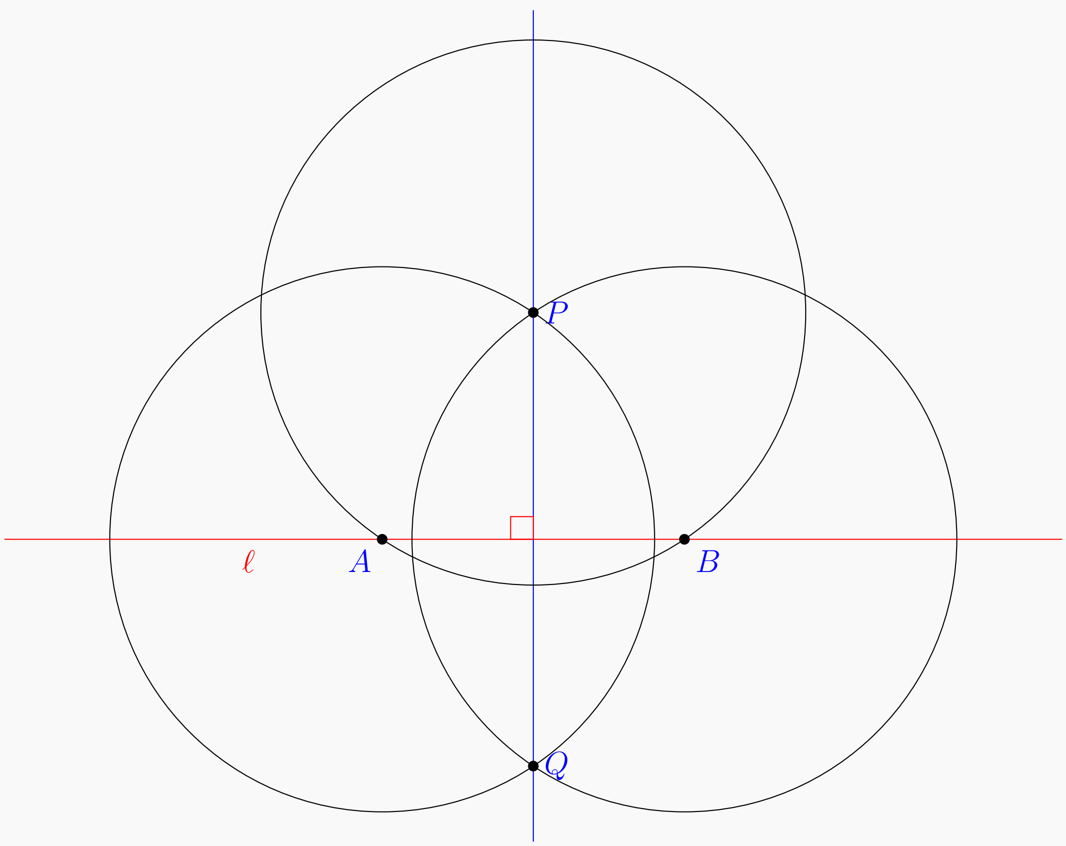
   (a) Construct a line passing through P that is perpendicular to ℓ.

   (b) Construct a line passing through P that is parallel to ℓ.

Solution:

(a) We can do this using a similar construction to the one for a perpendicular bisector (see lesson):

1. Construct a circle with center P and radius large enough that it intersects twice with ℓ.
2. Let A and B be the points of intersection of the circle with ℓ. Construct a circle with center A and radius AP, and a circle with center B and radius BP.
3. Let the second intersection of these two circles be Q. Then the line passing through P and Q is perpendicular to ℓ.

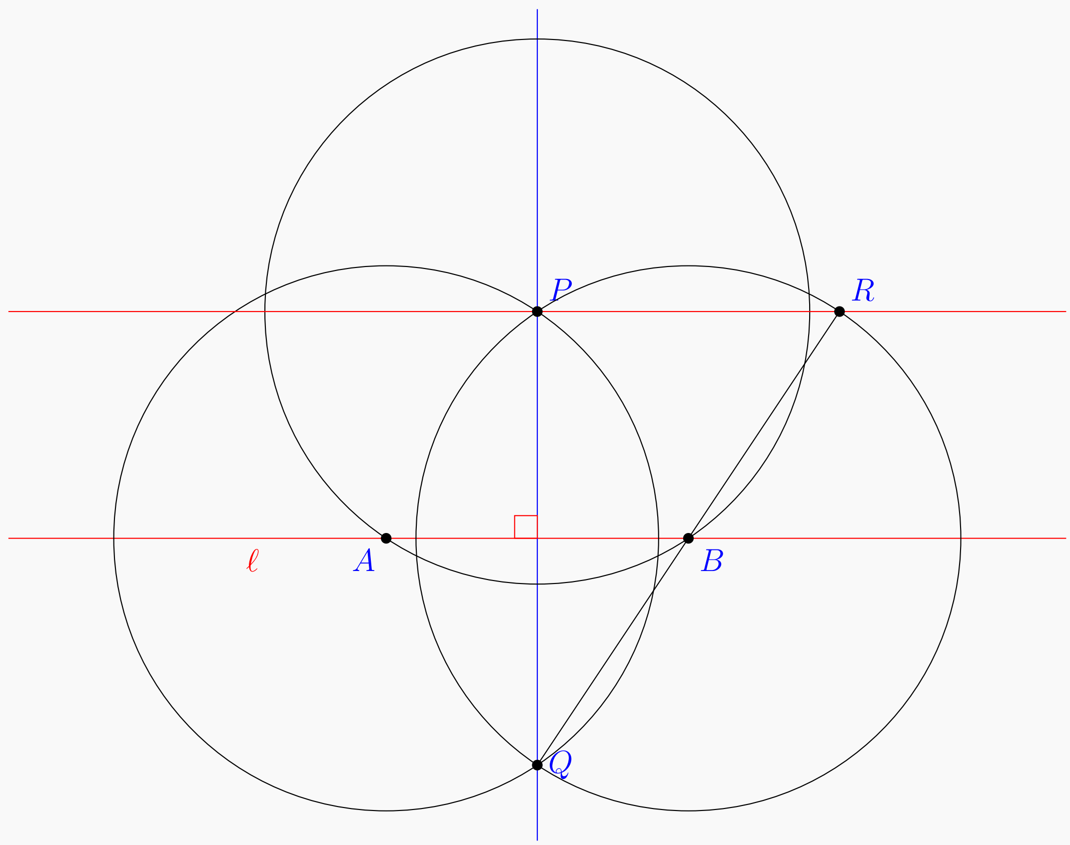


(b) The construction from part a) works even if P is on ℓ. Thus, we could technically use part a) twice:

1. Construct a line that passes through P and is perpendicular to ℓ.
2. Construct a line that passes through P and is perpendicular to the line you constructed in step 1. This newest line will be parallel to ℓ.

Here is an alternative solution:

1. Construct the line passing through P that is perpendicular to ℓ (as in part a).
2. Construct the line that passes through Q and B. Let it intersect the circle with center B and radius $BP = BQ$ at R.
3. The line passing through P and R is parallel to ℓ.



Bonus Exercise: why does this construction work? (see solution to question 8)

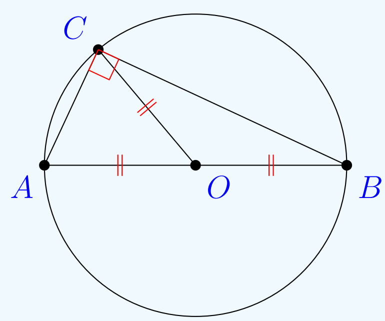
4. Construct a right triangle and its circumcircle. What do you notice?

Solution: We can replicate the same steps we used in the lesson to construct a right triangle and its circumcircle:

1. Construct points A and B, and construct the perpendicular bisector $\overline{AB}$.
2. Let C be the midpoint of $\overline{AB}$ and let D be a point (any point) on the perpendicular bisector of $\overline{AB}$. Then triangle ACD is a right triangle.
3. Construct the perpendicular bisectors of $\overline{AC}$ and $\overline{CD}$, and let them intersect at O, which is the circumcenter of triangle ACD.
4. Draw the circumcircle (center at O and radius AO).

Notice that O is the midpoint of $\overline{AC}$. In general, the circumcenter of any right triangle is always the midpoint of the **hypotenuse** (the long side of the right triangle). This occurs due to the following property of circles:

Let AB be the diameter of a circle and let C be a point on the same circle. Then $\angle ACB = 90°$:

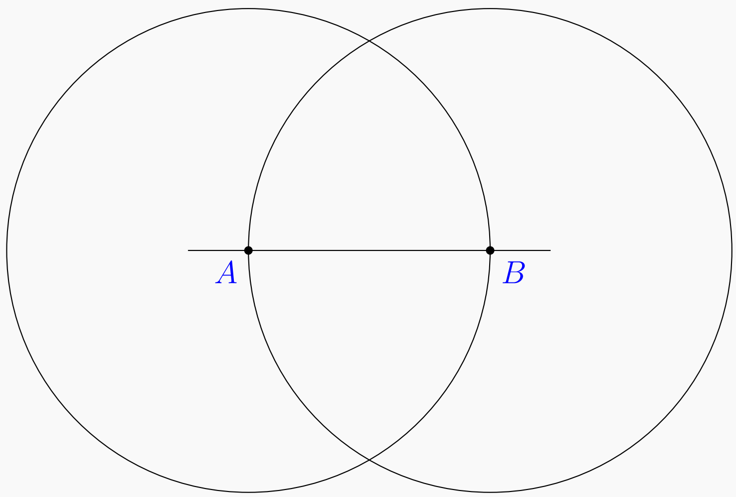


Hence, there is a much easier solution to this problem: construct a circle $\mathcal{C}$ with center O and let $\overline{AB}$ be a diameter of $\mathcal{C}$ (i.e., it passes through O). Let C be a point on $\mathcal{C}$ and now triangle ABC is a right triangle (since $\angle ACB = 90°$ by the above theorem), and $\mathcal{C}$ is its circumcircle.

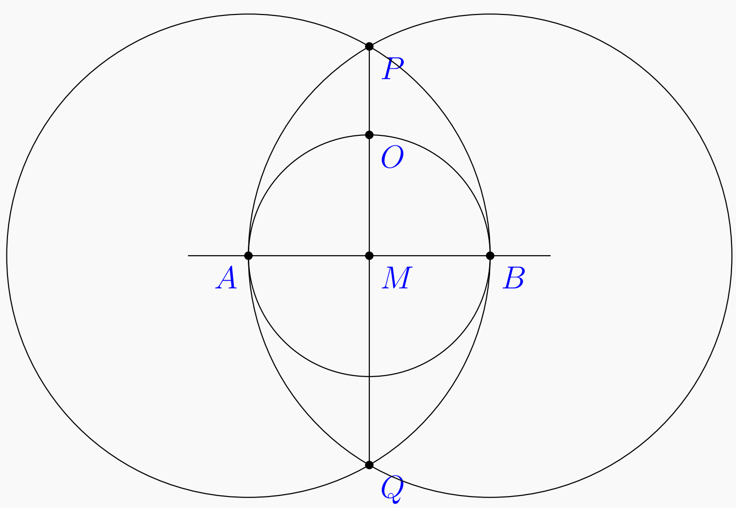
5. Construct a circle inscribed in a square.

Solution: There are many different ways of doing this, especially when using some of the constructions we've created in the lesson and in the first four problems. Here is one possible construction when starting from scratch:

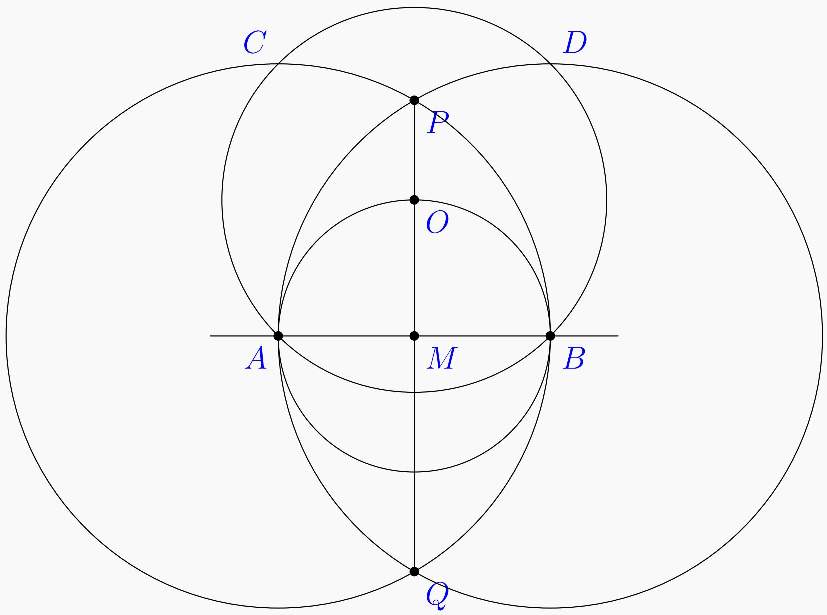
1. Construct a line segment $\overline{AB}$ and two circles of radius AB, with center A and B.



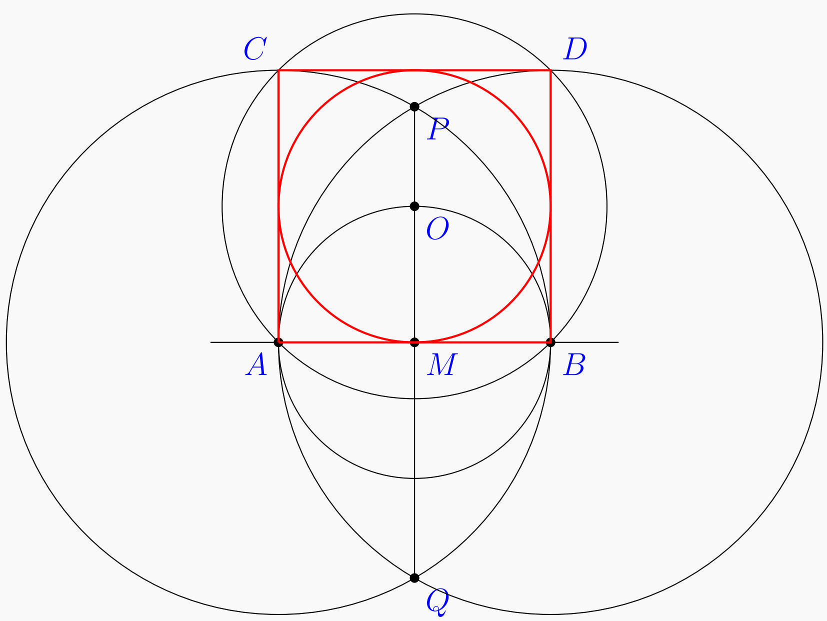
2. Let P and Q be the two points of intersections of the circles, let $\overline{PQ}$ intersect $\overline{AB}$ at M, and draw a circle with center M and radius AM, which intersects $\overline{PQ}$ at O:



3. Draw a circle with center O and radius OA. Let this intersect the two original circles at C and D, as shown below.



4. Finally, $ABCD$ is a square. Draw the circle with center O and radius OM. This will be inscribed in square $ABCD$.

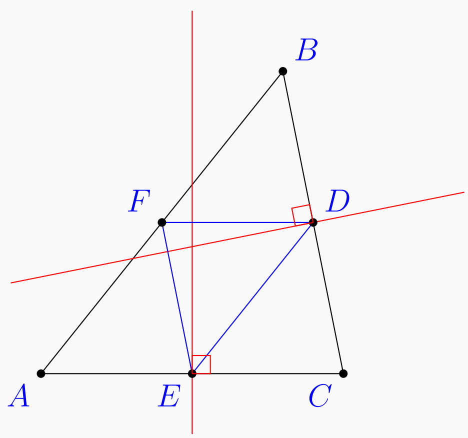


6. Draw a triangle and construct a circle which passes through the midpoints of each side. This circle is called the nine-point circle. Can you find any other 'interesting' points which lie on this circle?

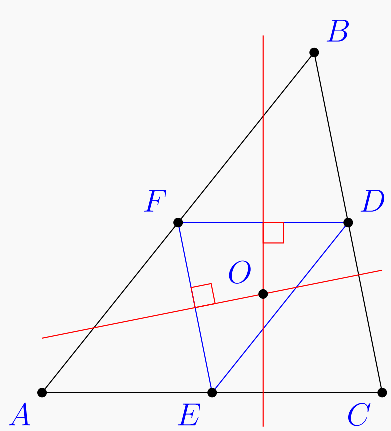
   The ***midpoint*** *of a line segment is the point on the segment which is halfway between the two endpoints.*

Solution: The actual construction can be done using the tools we learned in the lesson:

1. Identify the midpoints of each side. Connect them to form a smaller triangle.

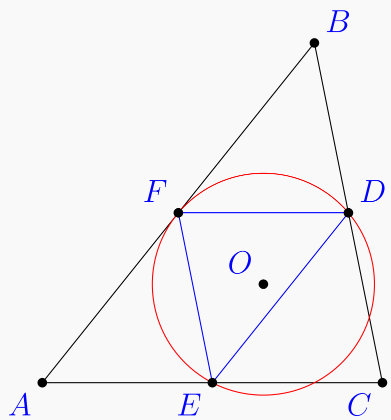


2. Construct two of the perpendicular bisectors of the smaller triangle. These intersect at the smaller triangle's circumcenter.

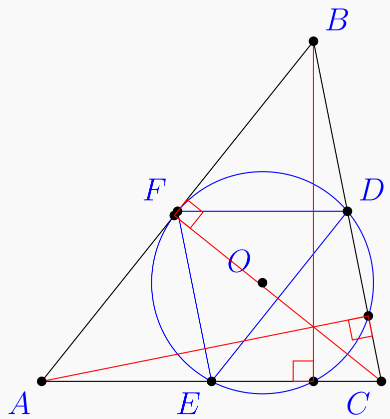


Note, if we were to fully draw every circle, the construction would get very messy. Instead, we can often get away with drawing very small part of the circle (instead of the full circle).

3. Finally, using this circumcenter, draw the nine-point circle.



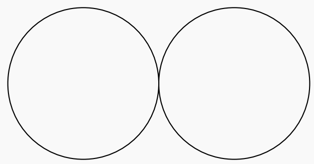
An altitude is a line segment which connects a vertex to its opposite side, such that it is perpendicular to the opposite side. Adding the altitudes to the above diagram:



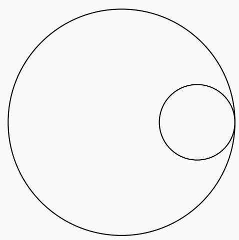
we see that the nine-point circle also passes through the **base** of each altitude. This is just one of many cool properties of this circle.

7. Two circles are tangent to each other if they intersect at exactly one point. Draw a point P. Construct 5 circles passing through P which are all tangent to each other.
Hint: Draw a circle with center O and radius OP. Let X be a point on $\overline{OP}$. Draw a circle with center X and radius XP. What do you notice?

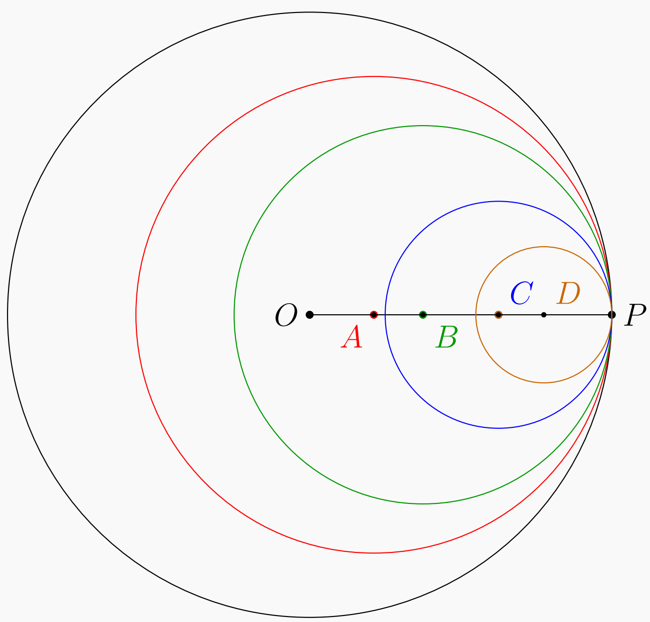
Solution: Two circles can be either **internally** tangent or **externally** tangent. For example, here are two circles which are externally tangent:



and here are two circles which are internally tangent:



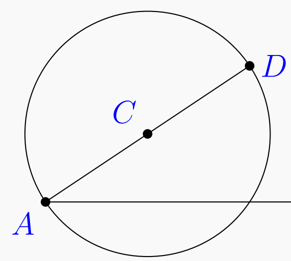
To make this construction, we use the hint: construct a circle with center O and radius OP. Let A, B, C, D be points on $\overline{OP}$. Construct the circles with centers A, B, C, and D, and with radius AP, BP, CP, and DP (respectively). Then we obtain



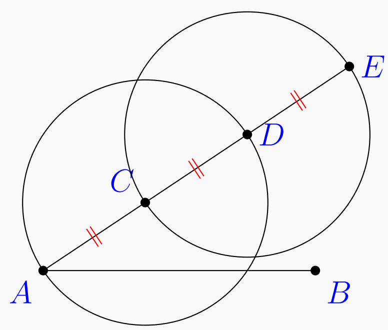
8. First, draw a line segment $\overline{AB}$. Trisect the line segment $\overline{AB}$ (separate it into three equal parts).

Solution: This is a tricky question. Here's one possible construction:

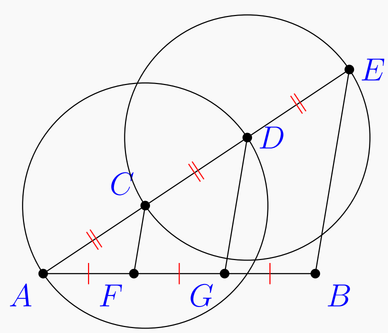
1. Let C be a point not on $\overline{AB}$ (see below). Construct a circle with center C and radius AC. Draw the line passing through A and C, and let it intersect the circle at D.



2. Construct a second circle with center D and radius CD (the same radius as the first) and let it intersect the line passing through A and C at E. Notice that C and D trisect $\overline{AE}$ ($AC = CD = DE$).



3. Finally, use question 3b) to construct lines passing through C and D which are parallel to $\overline{EB}$. These two lines will trisect $\overline{AB}$.



The reason why this works is a concept called similar triangles.

Two triangles are called **similar** if their angles are equal. Similar triangles are just the same triangle scaled differently. That is, if triangle ABC is similar to triangle DEF, with $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$, then

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

In our construction, ABE is similar and exactly three times bigger than ACF (since $AE = 3 \times AF$). Therefore, $AF = \frac{AB}{3}$. In the same way we can also prove that $AG = \frac{2AB}{3}$. Thus, F and G trisect $\overline{AB}$.